Taking the square and third root of a number

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Taking the square root of a number can be done easily on a piece of paper. The method is explained and an example for the integer number 71503936 is given.

Step 1: divide the number into groups of two digits, starting at the end: $\sqrt{71|50|39|36}$

Step 2: find the highest digit so that its square is not above the value of the group. In the example this digit is a '8' and its square is 64. (Trying '9' would result in 81, which is too high.)

Step 3: subtract that square from the first group (71−64 = 7) and use the next group (50) as suffix:

$$\begin{array}{l} \sqrt{71|50|39|36} = 8 \\ \underline{-\ 64\qquad\qquad\qquad} \\ \quad 7\ 50 \end{array}$$

Step 4: use the **estimator** $E = (2a \bullet \times \bullet)$ and find the same highest digit at all positions marked '•' so that the result for E is not above the remainder of the previous step (750), where a = 8 is the result so far and 2a is written as a one or two digit number, in this case 16.

In the example a = 8. The estimator then reads: $E = (16 \bullet \times \bullet)$. When we try '4' as the new digit we obtain: $E = (164 \times 4) = 656$. (Trying '5' would give a result above 750.)
Suffix the newly found digit '4' to the result that now becomes: a = 84 (so 2a = 168).

Step 5: subtract the estimator result 656 from the remainder 750 (resulting in 94) and suffix it with the next group (39) to obtain 9439 as the next remainder.

Repeat step 4, with $E = (168 \bullet \times \bullet)$ and the remainder 9439. Trying out digits will now give '5' as the highest possible, with $E = (1685 \times 5) = 8425$.
The '5' is suffixed to the result: a = 845 (2a = 1690).

Repeat step 5, subtracting 8425 from 9439 giving 1014, which in turn has to be suffixed with the last group (36) to give the remainder 101436.
Repeat step 4 with, $E = (1690 \bullet \times \bullet)$ where a '6' gets us $E = (16906 \times 6) = 101436$.
Subtracting this from the remainder gives 0 and there is no next digit group to suffix.
The '6' is suffixed to the result so that a = 8456, which is the exact solution of the square root.

The whole process is shown below. Note that the first time a = 0. In each estimator at the left the new digit is underlined.

$$\begin{array}{ll} & \sqrt{71|50|39|36} = 8456 \\ 8\text{x}8 & \underline{-\ 64} \\ & \quad 7\ 50 \\ 164\text{x}4 & \ \underline{-\ 6\ 56} \\ & \qquad 94\ 39 \\ 1685\text{x}5 & \quad \underline{-\ 84\ 25} \\ & \qquad 10\ 14\ 36 \\ 16906\text{x}6 & \quad \underline{-10\ 14\ 36} \\ & \qquad\qquad\qquad 0 \end{array}$$

For non-integer numbers the process is the same. The digit grouping starts at the decimal point and extends to the left and to the right, where the number is padded with zeros to the right, if needed.
Example: 69234.58761 is split into groups as 6|92|34|.58|76|10

Taking the third root of a number can also be done on a piece of paper without much difficulty, although the estimator is a bit more challenging. The method is similar to taking a square root and is explained with an example for the integer number 15069223.

Step 1: divide the number into groups of three digits, starting at the end: $\sqrt[3]{15|069|223}$

Step 2: find the highest digit so that its third power is not above the value of the group. In the example this digit is a '2' and its third power is 8. (Trying '3' would result in 27, which is too high.)

Step 3: subtract that third power from the first group (15−8 = 7) and use the next group (069) as suffix:

$$\sqrt[3]{15|069|223} = 2$$
$$\underline{-\ 8\qquad\qquad\qquad}$$
$$7\,069$$

Step 4: use the **estimator** $E = \left((300a^2 + 3a\bullet\times\bullet)\times\bullet\right)$ and find the same highest digit at all positions marked '•' so that the result for ***E*** is not above the remainder of the previous step (7069), where ***a*** is the result so far and 3a is written as a one or two digit number, in this case 6.

In the example ***a*** = 2. The estimator then reads: $E = \left((1200 + 6\bullet\times\bullet)\times\bullet\right)$. When we try '4' as the new digit we obtain: $\left((1200 + 64\times 4)\ \times\ 4\right) = 5824$. (Trying '5' would give a result above 7069.)

Step 5: subtract the estimator result 5824 from the remainder 7069 (resulting in 1245) and suffix it with the next (in this case the last) group (223) to obtain 1245223 as the next remainder.

Repeat step 4, now with ***a*** = 24 (this is again the result so far) to obtain:
$E = \left((172800 + 72\bullet\times\bullet)\times\bullet\right)$ and try to find the next digit.

Putting in a '7' at the position of the dots gives:
$E = \left((172800 + 727\times 7)\times 7\right) = 1245223$. Subtracting this from the result gives a remainder of 0 and there is no next group to use as a suffix, so we have an exact solution of 247 for the third power root of 15069223.

The whole process is shown below, where 8, 5824 and 1245223 are the estimator results for finding the three digits 2, 4 and 7 that form the result.
Note that the estimator also holds for the first digit which was '2' (since ***a*** = 0 at the start).

	$\sqrt[3]{15\|069\|223} = 247$
(2x2)x2	− 8
	7 069
(1200+64x4)x4	− 5 824
	1 245 223
(172800+727x7)x7	− 1 245 223
	0

For non-integer numbers the process is the same. The digit grouping starts at the decimal point and extends to the left and to the right, where the number is padded with zeros to the right, if needed.
Example: 69234.5876 is split into groups as 69|234|.587|600

November 2018